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Diffusion-limited aggregation and the Saffman-Taylor problem

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A random-walk model is proposed to simulate Darcy-law two-dimensional flows in hydrodynamics in the limit of zero-surface tension. The simulation is compared with the analytic and numerical results of the latter in steady-state and dynamic cases, respectively. The instabilities of the model in a flat interface are studied in the linear region. It is clear that the mean-field limit of diffusion-limited aggregation is a Saffman-Taylor problem.

Recently, there has been increasing interest in nonequilibrium dynamic problems, for example, models for dendritic growth[1] and diffusion-limited aggregation (DLA).[2] Another simple example is the Saffman-Taylor problem:[3] the interfacial motion of two incompressible fluids with different viscosities in a two-dimensional channel (Hele-Shaw cell). In the limit that the viscosity of one fluid and the interfacial surface tension go to zero, the equations governing the hydrodynamic motion here are very simple indeed. Since in the fluid with zero viscosity there is no friction inside the fluid and from the container, the pressure field is constant therein. In the viscous fluid, on the other hand, the velocity of the fluid is proportional to the pressure gradient because of the friction from the third dimension (from the plates containing the fluids). Let P be the pressure field in the viscous fluid, the incompressibility implies that P satisfies Laplace's equation:

$$\nabla^2 P = 0 \ . \tag{1a}$$

The continuity of pressure at the interface gives the boundary condition for P there:

$$P|_{\text{interface}} = \text{const} = 0 \ . \tag{1b}$$

Assume that the channel is in the $x-y$ plane and in the x direction. At two sides of the channel ($y = \pm y_0$), the velocity of the fluid must be parallel to the side; that is,

$$\left.\frac{\partial P}{\partial y}\right|_{y=\pm y_0} = 0 \ . \tag{1c}$$

The last boundary condition for P is that far away from the interface

$$P|_{x\to\infty} = -\frac{U_\infty}{\kappa} x \ , \tag{1d}$$

where U_∞ is the velocity of the viscous fluid far away from the interface and κ some positive constant containing the viscosity and the (z direction) thickness of the channel (permeability). The interface will move according to

$$V_n = -\kappa \nabla_n P|_{\text{interface}} \ , \tag{1e}$$

where n is the normal of the interface.

Equations (1a)–(1e) are all we need to describe the interfacial motion. Interestingly, Witten and Sander,[2] when deriving a mean-field equation for their DLA, got the same set of equations. It has been pointed out recently by Paterson[4] that a similarity exists between the interfacial motion of two fluids in porous media, which is also described by the above set of equations, and DLA. He compared his DLA simulations with some experiments of fluids in porous media. Kadanoff[5] pointed out that there is a connection between the Saffman-Taylor problem and DLA, while Nittmann, Daccord, and Stanley[6] did some experiments and DLA simulations of fluids in a Hele-Shaw cell in the zero-surface tension case where they concentrated on the fractal structures. In this paper, we study quantitatively the other limit—the mean-field limit—and show that the mean-field limit (MFL) of DLA is exactly the Saffman-Taylor problem.

The model we propose is very simple and somewhat different from DLA. We use the probability distribution p of random walkers to simulate the solution of Laplace's equation. Concerning Eq. (1), we make the substitution $p \to -P$. The simulation takes place in a channel of a two-dimensional lattice. Each lattice site is either "occupied" or "unoccupied." The occupied region represents the zero-viscosity fluid and the unoccupied the viscous one. An integer P is stored at each nearest neighbor to the occupied region. Starting with some initial interface, a random walker is released in the unoccupied region far away from the interface [refer to Eq. (1d)]. It walks randomly [refer to Eq. (1a)] until it reaches an occupied site. When this happens, the random walker is removed [refer to Eq. (1b)] and the last unoccupied site it visited will be registered once; i.e., its P value will be increased by 1. Another random walker then is released far away and starts a random walk in the unoccupied region until it is removed and some unoccupied site is registered. The process continues like this. When an unoccupied site has been registered M times, it becomes occupied [refer to Eq. (1e)]. The boundary condition on the walls [Eq. (1c)] is realized by letting a walker which hits the wall be reflected.

For $M = 1$, the model is DLA. When M goes to infinity, the model should be the MFL of DLA. Unfortunately, the solution of Eqs. (1) with negative P (positive p), which is the case when the nonviscous fluid is pushing the viscous one and is the original Saffman-Taylor problem, is unstable.[3] Furthermore, the instabilities will lead to singularities.[7,8] Hence, it is very hard (if not impossible) to simulate the hydrodynamic motion in this case or equivalently, to find the MFL of DLA in discrete latice. Now, let us look at Eqs. (1) more carefully. Notice that the transformation $P \to -P$ is identical to the time-reversal transformation. As in quantum-field theory, an antiparticle moving forward in time is identical to the particle moving backward in time.

Moreover, we know that if a solution of a differential equation is unstable in time it is then stable backward in time. So we can use antirandom walkers to get the time-reversed pictures of the Saffman-Taylor problem and DLA. Experimentally, this corresponds to switching the pressure gradient and therefore the viscous fluid pushing a nonviscous one. The algorithms described in the previous paragraph are all applicable, except that now the registrations are in the boundary of the occupied region (when an antirandom walker hits some occupied site, the site is registered once) and when an occupied site has had M registrations it becomes unoccupied (annihilated). Since this antiparticle simulation is stable, the discreteness of the lattice manifests itself as little as possible. Hence, a relatively small lattice channel is enough to get rather nice pictures. In fact, most of our simulations are done in a channel of 64-lattice-site width ($W=64$).

The steady-state motion of the Saffman-Taylor problem has been solved analytically,[3] which is a finger of the nonviscous fluid penetrating a viscous one. There exists a family of solutions with different finger widths and the shape of finger tip depends on its width.[3] The time reversal of these solutions are also steady-state motions and we can simulate them easily. We start with a finger of occupied region (nonviscous fluid) of certain width and arbitrary tip shape, and then do an anti-random-walker simulation. Soon we get a steady-state motion (the tip shape will not change any more). In Fig. 1 are shown simulations with different finger widths together with the analytic results.

Shariman and Bensimon[7] studied the dynamics of Eq. (1) numerically (Fig. 2). They found that a small perturbation (c) on a flat interface will develop a cusp (singularity) (a) within finite time. In Fig. 2 is also shown our simulation result in excellent agreement with their dynamic picture. We start from (a), and by annihilating the occupied region (the region under the curve) by antiparticles the interface moves to (b) and (c) successively.

In order to have a better understanding of the continuum limit of the model, we study, in the large M limit, the instabilities of a flat interface under an infinitesimal perturbation.[9] Imagine an infinite two-dimensional lattice in the x-y plane with lattice spacing a. We are interested in the structure on a length scale much much larger than a. Suppose initially, that a flat interface lies at $x=0$. The region $x\leq 0$ is occupied, and the region $x>0$ unoccupied. The probability distribution of random walkers satisfies the lattice version of Laplace's equation,

$$p(x+a,y)+p(x-a,y)+p(x,y+a)+p(x,y-a)-4p(x,y)=0 \quad . \tag{2a}$$

With the boundary condition on the inteface,

$$p(0,y)=0 \quad , \tag{2b}$$

and the boundary condition far away from interface,

$$p(x,y)|_{x\to\infty}=Cx \quad , \tag{2c}$$

where C is a constant. In this model we can arbitrarily define some Monte Carlo time unit, for example, $1/a^2$ random-walker steps. Then we can talk about the velocity of the interfacial growth. Notice that the random-walker flux through some bond in each step is

$$\tfrac{1}{4}[p(\mathbf{r}+a\hat{\mathbf{n}})-p(\mathbf{r})] \quad ,$$

where $\hat{\mathbf{n}}$ is the unit vector of the x or y direction. Hence, the flux in the unit time is given by

$$\frac{1}{a^2}\frac{1}{4}[p(\mathbf{r}+a\hat{\mathbf{n}})-p(\mathbf{r})]=\frac{1}{4a}\tilde{\nabla}_n p(\mathbf{r}) \quad ,$$

where $\tilde{\nabla}_n$ is the lattice gradient defined as above. So in large scale, the motion of the interface satisfies

$$V_n=\frac{a}{M}\frac{1}{4a}\tilde{\nabla}_n p(\mathbf{r})\Big|_{x=0}=\frac{1}{4M}\tilde{\nabla}_n p(\mathbf{r})\Big|_{x=0} \quad . \tag{2d}$$

The solution of Eqs. (2) is simply $p=Cx$ and $V_x=C/4M=V$. Notice that in the limit $a\to 0$, Eqs. (2) reduce to Eqs. (1).

Now let us introduce a small perturbation in the flat interface such that the perturbed interface is given by

$$x_l=Ae^{iky+\sigma t_M} \quad , \tag{3}$$

where $A\gg a$ and $kA\ll 1$. The solution of Eqs. (2a) and (2c), which vanishes at the interface Eq. (3) to the first or-

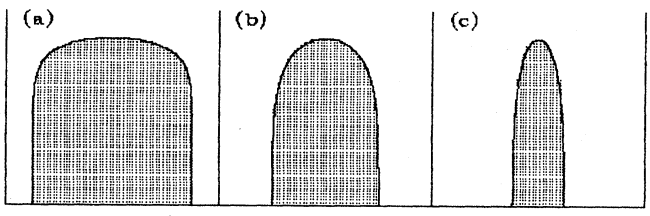

FIG. 1. Comparison of the steady-state interfacial motion between the simulation (dots) and the analytic solution (solid line), with the finger width being (a) 0.75, (b) 0.5, and (c) 0.25 of the channel width. In the simulation, the channel width $W=64$ and the MF number $M=20$.

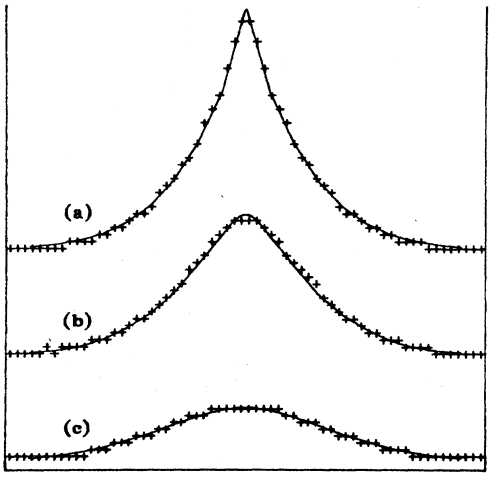

FIG. 2. Comparison of the interfacial dynamics betwen the simulation (+) and the numerical solution of Eqs. (1) (solid lines). The initial condition is (a) for the former and (c) for the latter. In the simulation, $W=64$ and $M=100$.

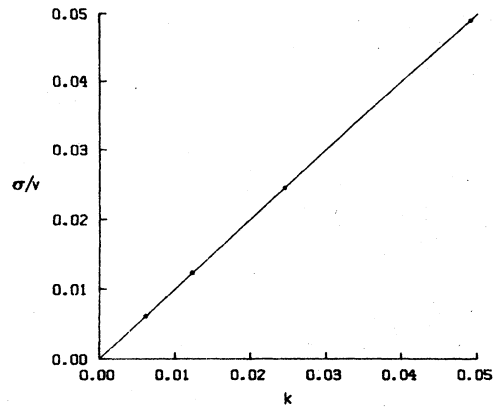

FIG. 3. Growth rate σ of an infinitesimal sinusoidal perturbation on a flat interface vs the wave number k of the perturbation. The soid line is Eq. (7) and the dots are simulation measurements.

der of kA, is

$$p = Cx - CA \exp(iky - qx + \sigma t_M) \ , \tag{4}$$

where k and q satisfy

$$\cos ka + \cosh qa - 2 = 0 \ . \tag{5}$$

Since $ka << 1$, $qa << 1$ and $k = q$. The velocity of the interfacial motion seen from the moving frame of the unperturbed interface is

$$v_x = \frac{\partial x_i}{\partial t_M} = A\sigma e^{iky + \sigma t_M}$$

$$= \frac{1}{4M} \tilde{\nabla}_x p\,(x,y) \Big|_{x = x_i} = VkAe^{iky + \sigma t_M} \ ,$$

where the last equality is to the first order of kA. So we end up with

$$\sigma = Vk \ . \tag{6}$$

In the corresponding simulation it is much easier to use S, the displacement of the unperturbed interface, as the measurement of time. Since $t_M = S/V$, we should be able to get

$$\sigma / V = k \ . \tag{7}$$

Figure 3 is a plot of the measurement with anti-random-walker simulation. It fits Eq. (7) perfectly.

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