

## Phases of Josephson Junction Ladders

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We study a Josephson junction ladder in a magnetic field in the absence of charging effects via a transfer matrix formalism. The eigenvalues of the transfer matrix are found numerically, giving a determination of the different phases of the ladder. The spatial periodicity of the ground state exhibits a devil's staircase as a function of the magnetic flux filling factor  $f$ . If the transverse Josephson coupling is varied, a continuous superconducting-normal transition in the transverse direction is observed, analogous to the breakdown of the Kolmogorov-Arnold-Moser trajectories in dynamical systems.

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Two-dimensional arrays of Josephson junctions have attracted much recent theoretical and experimental attention [1]. Interesting physics arises as a result of competing vortex-vortex and vortex-lattice interactions. It is also considered to be a convenient experimental realization of the frustrated XY models. In this paper, we discuss the simplest such system, namely, the Josephson junction ladder (JL) [2-4], shown in the inset of Fig. 1.

To construct the system, superconducting elements are placed at the ladder sites. Below the bulk superconducting-normal transition temperature, the state of each element is described by its charge and the phase of the superconducting wave function [5]. In this paper we neglect charging effects, which corresponds to the condition that  $4e^2/C \ll J$ , with  $C$  being the capacitance of the element and  $J$  the Josephson coupling. Let  $\theta_j$  ( $\theta'_j$ ) denote the phase on the upper (lower) branch of the ladder at the  $j$ th rung. The Hamiltonian for the array [6] can be written in terms of the gauge invariant phase differences,  $\gamma_j = \theta_j - \theta_{j-1} - (2\pi/\phi_0) \int_{j-1}^j A_x dx$ ,  $\gamma'_j = \theta'_j - \theta'_{j-1} - (2\pi/\phi_0) \int_{j-1}^j A_x dx$ , and  $\alpha_j = \theta'_j - \theta_j - (2\pi/\phi_0) \int_j^{j+1} A_y dx$ :

$$\mathcal{H} = - \sum_j (J_x \cos \gamma_j + J_x \cos \gamma'_j + J_y \cos \alpha_j), \quad (1)$$

where  $A_x$  and  $A_y$  are the components of the magnetic vector potential along and transverse to the ladder, respectively, and  $\phi_0$  the flux quantum. The sum of the phase differences around a plaquette is constrained by  $\gamma_j - \gamma'_j + \alpha_j - \alpha_{j-1} = 2\pi(f - n_j)$ , where  $n_j = 0, \pm 1, \pm 2, \dots$  is the vortex occupancy number and  $f = \phi/\phi_0$  with  $\phi$  being the magnetic flux through a plaquette. With this constraint, it is convenient to write Eq. (1) in the form

$$\mathcal{H} = - J \sum_j \{ 2 \cos \eta_j \cos [(\alpha_{j-1} - \alpha_j)/2 + \pi(f - n_j)] + J_t \cos \alpha_j \}, \quad (2)$$

where  $\eta_j = (\gamma_j + \gamma'_j)/2$ ,  $J = J_x$ , and  $J_t = J_y/J_x$ . The Hamiltonian is symmetric under  $f \rightarrow f + 1$  with  $n_j \rightarrow n_j + 1$ , and  $f \rightarrow -f$  with  $n_j \rightarrow -n_j$ , thus it is sufficient to study only the region  $0 \leq f \leq 0.5$ . Since in one dimension ordered phases occur only at zero temperature, the main interest is in the ground states of the ladder and the low temperature excitations. Note that in Eq. (2)  $\eta_j$  decouples from  $\alpha_j$  and  $n_j$ , so that all the ground states have  $\eta_j = 0$  to minimize  $\mathcal{H}$ . The ground states will be among the solutions to the current conservation equations

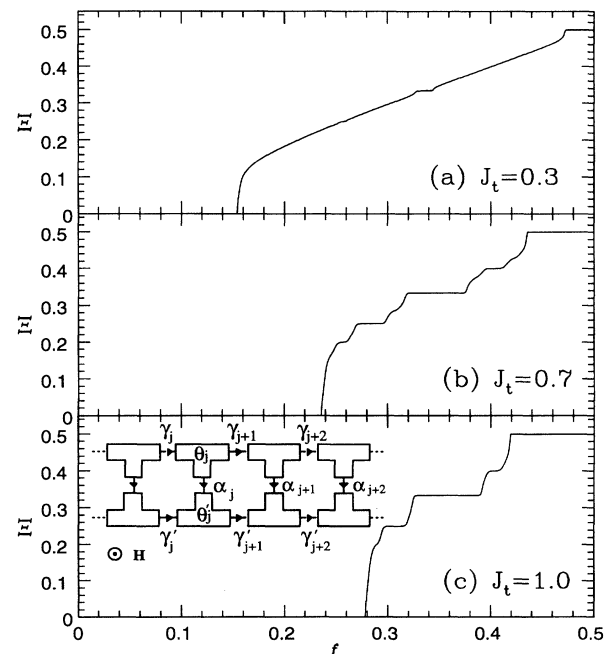


FIG. 1. Periodicity  $\Xi = \text{Arg}(\lambda_1)/2\pi$  vs  $f$  for  $k_B T/J = 0.005$  and (a)  $J_t = 0.3$ , (b)  $J_t = 0.7$ , and (c)  $J_t = 1.0$ . Inset: The Josephson junction ladder is formed by the arrangement of the superconducting islands. The field  $\mathbf{H}$  is out of the page and the arrows indicate the direction of the gauge invariant phase differences.

$\partial \mathcal{H} / \partial \alpha_j = 0$ :

$$J_t \sin \alpha_j = \sin[(\alpha_{j-1} - \alpha_j)/2 + \pi(f - n_j)] \\ - \sin[(\alpha_j - \alpha_{j+1})/2 + \pi(f - n_{j+1})]. \quad (3)$$

For any given  $f$  there are a host of solutions to Eq. (3). The solution that minimizes the energy must be selected to obtain the ground state.

If one expands the  $\cos[(\alpha_{j-1} - \alpha_j)/2 + \pi(f - n_j)]$  term in Eq. (2) about its maximum, the discrete sine-Gordon (DSG) model is obtained. A vortex ( $n_j = 1$ ) in the JLL corresponds to a kink in the DSG. Kardar [2] used this analogy to argue that this system should show similar behavior to the DSG, which has been studied by several authors [8–10]. This analogy is only valid for  $J_t$  very small so that the interplaquette term dominates the behavior of the system making the expansion about its maximum a reasonable assumption. However, much of the interesting behavior of the DSG occurs in regions

of large  $J_t$  ( $J_t \sim 1$ ). Furthermore, much of the work by Aubry [8] on the DSG relies on the convexity of the coupling potential which we do not have in the JLL.

In this Letter we formulate the problem in terms of a transfer matrix obtained from the full partition function of the ladder. The eigenvalues and eigenfunctions of the transfer matrix are found numerically to determine the phases of the ladder as functions of  $f$  and  $J_t$ . We find that the spatial periodicity of the ground states goes through a devil's staircase as a function of  $f$ . We then study the properties of various ground states and the low temperature excitations. As  $J_t$  is varied, all incommensurate ground states are found to undergo a superconducting-normal transition at a certain  $J_t$  which depends on  $f$ . Finally, we discuss the critical current.

The partition function for the ladder, with periodic boundary conditions and  $K = J/k_B T$ , is

$$Z = \prod_i^N \int_{-\pi}^{\pi} d\alpha_i d\eta_i \sum_{\{n_i\}} \exp\{K(2 \cos \eta_i \cos[(\alpha_{i-1} - \alpha_i)/2 + \pi(f - n_i)] + J_t \cos \alpha_i)\}. \quad (4)$$

The  $\eta_i$  can be integrated out, resulting in a simple transfer matrix formalism for the partition function involving only the transverse phase differences:  $Z = \prod_i^N \int_{-\pi}^{\pi} d\alpha_i P(\alpha_{i-1}, \alpha_i) = \text{Tr} \hat{P}^N$ . The transfer matrix elements  $P(\alpha, \alpha')$  are

$$P(\alpha, \alpha') = 4\pi \exp[KJ_t(\cos \alpha + \cos \alpha')/2] \\ \times I_0(2K \cos[(\alpha - \alpha')/2 + \pi f]), \quad (5)$$

where  $I_0$  is the zeroth order modified Bessel function. Note that the elements of  $\hat{P}$  are real and positive, so that its largest eigenvalue  $\lambda_0$  is real, positive, and nondegenerate. However, since  $\hat{P}$  is not symmetric (except for  $f = 0$  and  $f = 1/2$ ), other eigenvalues can form complex conjugate pairs. As we will see from the correlation function, these complex eigenvalues determine the spatial periodicity of the ground states.

The two point correlation function of  $\alpha_j$ 's is

$$\langle e^{i(\alpha_0 - \alpha_l)} \rangle = \lim_{N \rightarrow \infty} \frac{[\prod_i^N \int_{-\pi}^{\pi} d\alpha_i P(\alpha_{i-1}, \alpha_i)] e^{i(\alpha_0 - \alpha_l)}}{Z} \\ = \sum_n c_n \left( \frac{\lambda_n}{\lambda_0} \right)^l, \quad (6)$$

where the  $\lambda_n$  are the eigenvalues ( $|\lambda_n| \geq |\lambda_{n+1}|$  and  $n = 0, 1, 2, \dots$ ) and the constants  $c_n = \int_{-\pi}^{\pi} d\alpha' \times \psi_0^L(\alpha') e^{i\alpha'} \psi_n^R(\alpha') \int_{-\pi}^{\pi} d\alpha \psi_n^L(\alpha) e^{-i\alpha} \psi_0^R(\alpha)$ . (Note that since  $\hat{P}$  is not symmetric both right  $\psi_n^R$  and left  $\psi_n^L$  eigenfunctions are needed.) If  $\lambda_1$  is real and  $|\lambda_1| > |\lambda_2|$ , Eq. (6) simplifies for large  $l$  to

$$\langle e^{i(\alpha_0 - \alpha_l)} \rangle = c_0 + c_1 \left( \frac{\lambda_1}{\lambda_0} \right)^l, \quad |\lambda_1| > |\lambda_2|.$$

If  $\lambda_1 = \lambda_2^* = |\lambda_1| e^{i2\pi\Xi}$ , Eq. (6) for large  $l$  is

$$\langle e^{i(\alpha_0 - \alpha_l)} \rangle = c_0 + (c_1 e^{i2\pi\Xi l} + c_2 e^{-i2\pi\Xi l}) \left| \frac{\lambda_1}{\lambda_0} \right|^l.$$

Note that while the correlation length is given by  $\xi = [\ln |\lambda_0/\lambda_1|]^{-1}$  the quantity  $\Xi = \text{Arg}(\lambda_1)/2\pi$  determines the spatial periodicity of the state. We found numerically [7] that for  $f$  smaller than a critical value  $f_{c1}$  which depends on  $J_t$ , both  $\lambda_1$  and  $\lambda_2$  are real. These two eigenvalues become degenerate at  $f_{c1}$ , and then bifurcate into a complex conjugate pair.  $\Xi$  as a function of  $f$  is shown in Fig. 1 for several different values of  $J_t$ . The shape of these curves is generally referred to as a devil's staircase. The steps of the staircase are at  $\Xi = p/q$ , where  $p$  and  $q$  are integers. These are commensurate states with  $p$  vortices in each unit cell, which consists of  $q$  plaquettes. For small  $J_t$ , the flat steps are connected by smooth differentiable curves; most states on the  $\Xi - f$  curve are incommensurate states. As  $J_t$  increases, more and more steps appear and grow at the expense of the smooth regions. At  $J_t = J_t^c \approx 0.7$  the staircase becomes complete, i.e., there is a step for every rational  $\Xi$ , and the set of  $f$  which corresponds to irrational  $\Xi$  has zero measure. For  $J_t > J_t^c$ , the staircase becomes overcomplete, i.e., steps of lower order rationals grow and steps of higher order rationals disappear [11]. Another important characterization of a state is the phase density  $\rho(\alpha)$ :  $\rho(\alpha)d\alpha$  is the average fraction of all sites in the ladder with  $\alpha < \alpha_i < \alpha + d\alpha$ . If  $\rho(\alpha)$  is a smooth function for  $\alpha \in (-\pi, \pi]$  at  $T = 0$ , the ground state energy is invariant under an adiabatic change of  $\alpha$ 's. Consequently, there is no phase coherence between upper and lower branches of the ladder and hence no superconductivity in the transverse direction. In this case, we say that the  $\alpha$ 's are unpinned. If there exist finite intervals of  $\alpha$  on which  $\rho(\alpha) = 0$ , the  $\alpha$ 's are pinned and there will be phase coherence between the upper and lower

branches. In terms of the transfer matrix, the phase density is the product of the left and right eigenfunctions of  $\lambda_0$  [12],  $\rho(\alpha) = \psi_0^L(\alpha)\psi_0^R(\alpha)$ .

We first discuss the case where  $f < f_{c1}$ . These are the ‘‘Meissner’’ states in the sense that there are no vortices ( $n_i = 0$ ) in the ladder. The ground state is simply  $\alpha_i = 0$ ,  $\gamma_j = \pi f$ , and  $\gamma'_j = -\pi f$ , so that there is a global ‘‘screening’’ current  $\pm J_x \sin(\pi f)$  in the upper and lower branches of the ladder [2]. The phase density  $\rho(\alpha) = \delta(\alpha)$ . The properties of the Meissner state can be studied by expanding Eq. (2) around  $\alpha_i = 0$ :  $\mathcal{H}_M = (J/4) \sum_j [\cos(\pi f)(\alpha_{j-1} - \alpha_j)^2 + 2J_t \alpha_j^2]$ . The current conservation Eq. (3) becomes

$$\alpha_{j+1} = 2[1 + J_t/\cos(\pi f)]\alpha_j - \alpha_{j-1}. \quad (7)$$

Besides the ground state  $\alpha_j = 0$ , there are another two linearly independent solutions  $\alpha_j = e^{\pm j/\xi_M}$  of Eq. (7) which describe collective fluctuations about the ground state, where

$$\frac{1}{\xi_M} = \ln \left[ 1 + \frac{J_t}{\cos \pi f} + \sqrt{\frac{2J_t}{\cos \pi f} + \left(\frac{J_t}{\cos \pi f}\right)^2} \right]. \quad (8)$$

$\xi_M$  is the low temperature correlation length for the Meissner state. (Note that  $\xi_M < 1$  for  $J_t \sim 1$ , making a continuum approximation invalid.) As  $f$  increases, the Meissner state becomes unstable to the formation of vortices. A vortex is constructed by patching the two solutions of Eq. (7) together using a matching condition. The energy  $\epsilon_v$  of a single vortex is found to be

$$\epsilon_v \approx [2 + (\pi^2/8) \tanh(1/2\xi_M)] \times \cos(\pi f) - (\pi + 1) \sin(\pi f) + 2J_t,$$

for  $J_t$  close to 1. The zero of  $\epsilon_v$  determines  $f_{c1}$ , which is in good agreement with the numerical result from the transfer matrix. For  $f > f_{c1}$ ,  $\epsilon_v$  is negative and vortices are spontaneously created. When vortices are far apart their interaction is caused only by the exponentially small overlap. The corresponding repulsion energy is of the order  $J \exp(-l/\xi_M)$ , where  $l$  is the distance between vortices. This leads to a free energy per plaquette of  $F = \epsilon_v/l + J \exp(-l/\xi_M)/l$  [10]. Minimizing this free energy as a function of  $l$  gives the vortex density for  $f \rightarrow f_{c1}^+$ :  $\langle n_j \rangle = l^{-1} = [\xi_M \ln |f_{c1} - f|]^{-1}$ .

We now discuss the commensurate vortex states, taking the one with  $\Xi = 1/2$  as an example. This state has many similarities to the Meissner state but some important differences. The ground state is  $(\alpha_0, n_0) = (\arctan[(2/J_t) \sin(\pi f)], 0)$ ,  $(\alpha_1, n_1) = (-\alpha_0, 1)$ , and  $(\alpha_{i\pm 2}, n_{i\pm 2}) = (\alpha_i, n_i)$ , so that there is a global screening current in the upper and lower branches of the ladder of  $\pm 2\pi J(f - 1/2)/\sqrt{4 + J_t^2}$ . Global screening, which is absent in an infinite 2D array, is the key reason for the existence of the steps at  $\Xi = p/q$ . It is easy to see that the symmetry of this  $\Xi = 1/2$  vortex state is that of the (antiferromagnetic) Ising model.

The low temperature excitations are domain boundaries between the two degenerate ground states. The energy of the domain boundary  $J\epsilon_b$  can be estimated using similar methods to those used to derive  $\epsilon_v$  for the Meissner state. We found that  $\epsilon_b = \epsilon_b^0 - (\pi^2/\sqrt{4 + J_t^2})|f - 1/2|$ , where  $\epsilon_b^0$  depends only on  $J_t$ . Thus the correlation length diverges with temperature as  $\xi \sim \exp(2J\epsilon_b/k_B T)$ . The transition from the  $\Xi = 1/2$  state to nearby vortex states happens when  $f$  is such that  $\epsilon_b = 0$ ; it is similar to the transition from the Meissner state to its nearby vortex states. All other steps  $\Xi = p/q$  can be analyzed similarly. For comparison, we have evaluated  $\xi$  for various values of  $f$  and  $T$  from the transfer matrix and found that  $\xi$  fits  $\xi \sim \exp(2J\epsilon_b/k_B T)$  (typically over several decades) at low temperature. The value of  $\epsilon_b$  as a function of  $f$  is shown in Fig. 2 for  $J_t = 1$ . The agreement with the above estimate for the  $\Xi = 1/2$  step is excellent. The tips of the peaks in Fig. 2 for states with  $\Xi = 1/q$  fit the relationship  $\tau \sim \exp(-q/l_0)$  with  $l_0 \approx 0.77$ , which is in good agreement with  $\xi_M(f = 0) \approx 0.76$  of Eq. (8). The low temperature transverse resistance should be proportional to the thermal activation rate of domain walls:  $R_t \sim \exp(-2J\epsilon_b/k_B T)$  (with  $\epsilon_b$  replaced with  $\epsilon_v$  for the Meissner state), and so facilitates direct comparison with experiment [4].

We now discuss the superconducting-normal transition in the transverse direction. For  $J_t = 0$ , the ground state has  $\gamma_i = \gamma'_i = 0$  and

$$\alpha_j = 2\pi f j + \alpha_0 - 2\pi \sum_{i=0}^{j-1} n_i. \quad (9)$$

The average vortex density  $\langle n_j \rangle$  is  $f$ ; screening currents are absent.  $\alpha_0$  in Eq. (9) is arbitrary; the  $\alpha$ 's are unpinned for all  $f$ . The system is simply two uncoupled 1D XY chains, so that the correlation length  $\xi = 1/k_B T$ . The system is superconducting at zero temperature along the ladder, but not in the transverse direction. As  $J_t$  rises above zero we observe a distinct difference between the system at rational and irrational values of  $f$ . For  $f$  rational, the  $\alpha$ 's become pinned for  $J_t > 0$  [ $\rho(\alpha)$  is a finite sum of delta functions] and the ladder is

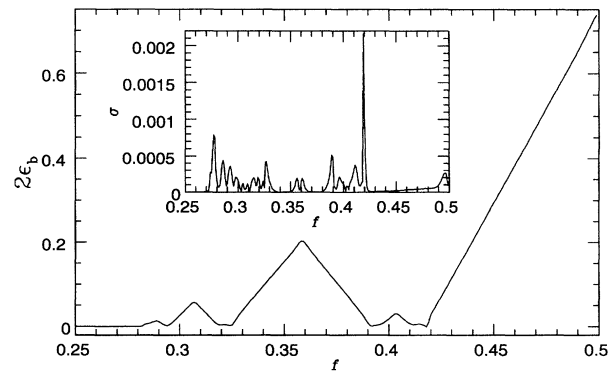


FIG. 2. Effective Ising coupling as a function of  $f$  for  $J_t = 1$ . Inset: Statistical error for  $2\epsilon_b$  in the fit versus  $f$ .

superconducting in *both* the longitudinal and transverse directions at zero temperature. The behavior for irrational  $f$  is illustrated in the following for the state with  $\Xi = a_g$ , where  $a_g = (3 - \sqrt{5})/2$ . Figure 3 displays  $\rho(\alpha)$  for several different  $J_t$  at  $\Xi = a_g$ . We see that the zero-frequency phonon mode [the smoothness of  $\rho(\alpha)$ ] persists for small  $J_t > 0$  until a critical value  $J_t^c(f) \approx 0.7$ , where the  $\alpha$ 's become pinned and the ladder becomes superconducting in the transverse direction. In the DSG, the pinning transition of this state coincides with the devil's staircase of Fig. 1 becoming complete [8,9]. (If the  $\alpha_j$ 's are pinned in this state, then all incommensurate states should be pinned.) The pinning transition of the incommensurate states can be also studied using Eq. (3), which can be viewed as a two-dimensional map. The disappearance of the zero-frequency phonon mode for irrational  $\Xi$ 's at finite small  $J_t^c(f)$  is equivalent to the breakdown of the Kolmogorov-Arnold-Moser (KAM) trajectories of the map [13].

We now turn to the subject of critical currents along the ladder. One can obtain an estimate for the critical current by performing a perturbation expansion (i.e.,  $\{n_j\}$  remain fixed) around the ground state and imposing the current constraint of  $\sin \gamma_j + \sin \gamma_j' = I$ . Let  $\delta \gamma_j$ ,  $\delta \gamma_j'$ , and  $\delta \alpha_j$  be the change of  $\gamma_j$ ,  $\gamma_j'$ , and  $\alpha_j$  in the current carrying state. One finds that stability of the ground state requires that  $\delta \alpha_j = 0$ , and consequently  $\delta \gamma_j = \delta \gamma_j' = I/2 \cos \gamma_j$ . The critical current can be estimated by the requirement that the  $\gamma_j$  do not pass through  $\pi/2$ , which gives  $I_c = 2(\pi/2 - \gamma_{\max}) \cos \gamma_{\max}$ , where  $\gamma_{\max} = \max_j(\gamma_j)$ . In all ground states we examined, commensurate and incommensurate, we found that  $\gamma_{\max} < \pi/2$ , implying a finite critical current for all  $f$ .

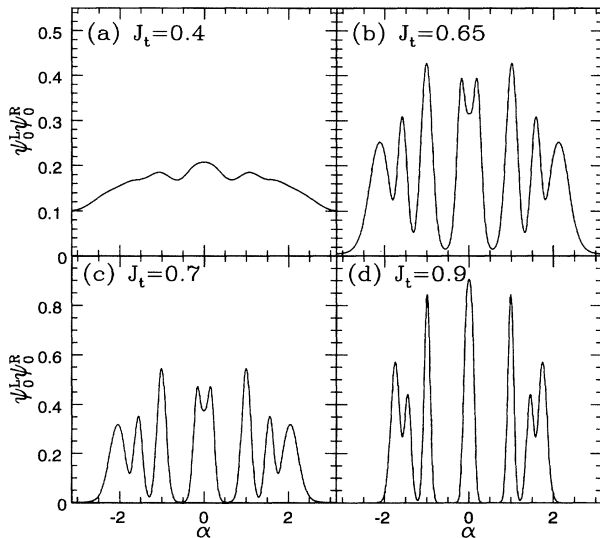


FIG. 3.  $\rho(\alpha) = \psi_0^L(\alpha)\psi_0^R(\alpha)$  vs  $\alpha$  at  $k_B T/J = 0.004$  and  $\Xi = 0.381966011\dots$ , and for (a)  $J_t = 0.4$ , (b)  $J_t = 0.65$ , (c)  $J_t = 0.7$ , and (d)  $J_t = 0.9$ . Note the smaller scale for the upper plots.

In conclusion, we have studied the equilibrium behavior of a Josephson junction ladder in a magnetic field in the absence of charging effects. Screening currents play an important role in this system, resulting in the spatial periodicity of the ground state climbing a devil's staircase as a function of  $f$ . Incommensurate states undergo a superconducting-normal transition in the transverse direction as  $J_t$  is increased, so that for  $J_t > J_t^c \approx 0.7$  the ladder is superconducting in both the longitudinal and transverse directions for all  $f$ . The critical current along the ladder is found to be finite for all  $f$ . Finally, although in one dimension there is no phase transition and long range order at finite temperature, our study showed that the correlation lengths in vortex states are extremely long for reasonably low temperatures. Thus one could test experimentally the predictions for the vortex configuration by, for instance, direct imaging via a scanning Hall-probe microscope or measuring the fractional giant Shapiro steps [14].

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