Fractal Dimension of Julia Set for Non-analytic Maps

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The Hausdorff dimensions of the Julia sets for non-analytic maps: $f(z) = z^2 + \epsilon z^*$ and $f(z) = {z^*}^2 + \epsilon$ are calculated perturbatively for small ϵ. It is shown that Ruelle's formula for Hausdorff dimensions of analytic maps can not be generalized to non-analytic maps.

I. INTRODUCTION

The Julia set J of a map is the closure of the unstable periodic points [1–4]. It is an invariant set of the map and is usually a "repeller", that is, points close to J will be repelled away by successive iterations of the map. A simple example is the map on the complex plane: $f(z) = z^2$, for which J is the unit circle. Points close to J will flow to one of the two stable fixed points: 0 and ∞. Thus J is the boundary or separator of basins of attraction. A much more complicated geometry appears for the Julia set of the map:

$$f(z) = z^2 + c, \tag{1}$$

where c is a non-zero constant (see Fig. 1(a) for an example and Ref. [4] for many other examples). In this case, J is a fractal and its topology undergoes drastic changes as c varies.

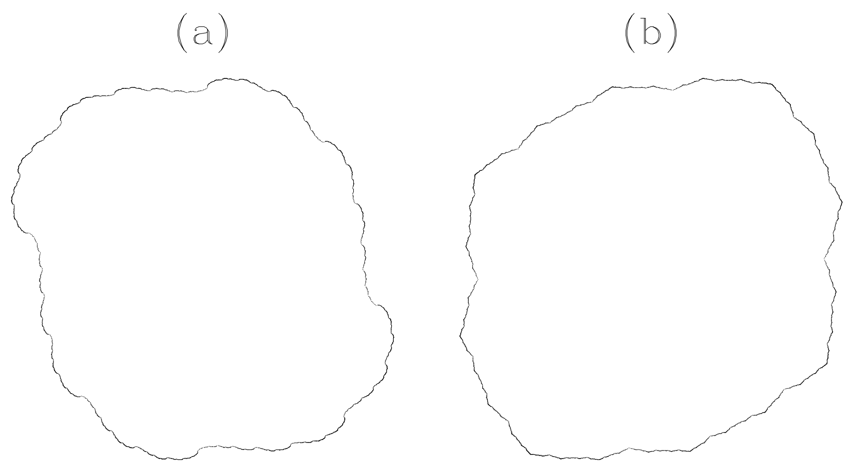

FIG. 1. The Julia set for (a) $f(z) = z^2 + \epsilon$ and (b) $f(z) = {z^*}^2 + \epsilon$, for $\epsilon = 0.15 + i0.15$.

Before we proceed further, let us define a few notations. We denote f^n to be n successive iterations of the map. That is $f^n(z) = f(f^{n-1}(z))$. The set of all unstable cycles of length n is denoted by $\mathrm{Fix}\, f^n$. Df is the derivative matrix of f. If f is an analytic map, i.e. $\partial u/\partial x = \partial v/\partial y$ and $\partial v/\partial x = -\partial u/\partial y$ with $f(z = x + iy) = u + iv$, then $\det Df = |df/dz|^2$.

For analytic maps, the Hausdorff dimension D_H of the Julia set J can be calculated with a formula due to a theorem of Ruelle [5]:

$$\lim_{n\to\infty} A_n(D_H) = 1, \tag{2}$$

where

$$A_n(D) = \sum_{z\in \mathrm{Fix} f^n} |\frac{df^n}{dz}|^{-D}. \tag{3}$$

Using the formula, Ruelle [5] and Widom *et al.* [6] calculated D_H for the map (1) in powers of c for small $|c|$. It was not clear then whether the formulas (2) and (3) can be generalized to non-analytic maps. The natural generalization of (3) to non-analytic maps would be

$$A_n(D) = \sum_{z \in \mathrm{Fix} f^n} |\det Df^n|^{-D/2}. \tag{4}$$

The calculations I present below show that the combination of (2) and (4) does not give the correct D_H for non-analytic maps in general and D_H can be calculated directly with the perturbation theory developed in Ref. [6].

II. THE MAP $f(z) = z^2 + \epsilon z^*$

Let us first consider the non-analytic map

$$f(z) = z^2 + \epsilon z^*, \tag{5}$$

where $*$ denotes the complex conjugate. When $\epsilon = 0$ the Julia set J is the unit circle and can be parametrized as $z(t) = e^{2\pi i t}$. The map on J is

$$f(z(t)) = z(2t). \tag{6}$$

When $\epsilon \neq 0$ but small enough so that J is topologically equivalent to a circle we can still parametrize J so that Eq. (6) is satisfied [2,3]. If a map f_ϵ with a parameter ϵ satisfies

$$[f_\epsilon(z)]^* = f_{\epsilon^*}(z^*), \tag{7}$$

then

$$z \in \mathrm{Fix}\, f_\epsilon^n \quad \Leftrightarrow \quad z^* \in \mathrm{Fix}\, f_{\epsilon^*}^n, \tag{8}$$

which implies that

$$J(f_\epsilon) = [J(f_{\epsilon^*})]^*, \tag{9}$$

where $J(f)$ is the Julia set of f. In particular, if J can be parametrized as $z(t)$ then

$$z_\epsilon(t) = z_{\epsilon^*}^*(-t). \tag{10}$$

It is easy to see that the map (5) satisfies Eq. (7).

Following Widom *et al.* [6], we formally expand $z(t)$ in powers of ϵ

$$z(t) = e^{2\pi i t}[1 + \epsilon U_1(t) + \epsilon^* \tilde{U}_1(t) + \epsilon^2 U_2(t) + {\epsilon^*}^2 \tilde{U}_2(t) + \epsilon\epsilon^* \hat{U}_2(t) + \cdots], \tag{11}$$

where the functions $U_1(t), \tilde{U}_1(t), U_2(t), \tilde{U}_2(t), \hat{U}_2(t), \ldots$ are all periodic with period 1. Eq. (10) implies that all the functions $U(t)$ satisfies $U(t) = U^*(-t)$. Substituting (11) into (6) and equating terms with the same power of ϵ, we get

$$U_1(2t) - 2U_1(t) = e^{-6\pi i t}, \tag{12}$$

$$\tilde{U}_1(2t) - 2\tilde{U}_1(t) = 0, \tag{13}$$

$$U_2(2t) - 2U_2(t) = U_1^2(t) + e^{-6\pi i t}\tilde{U}_1^*(t), \tag{14}$$

$$\tilde{U}_2(2t) - 2\tilde{U}_2(t) = \tilde{U}_1^2(t), \tag{15}$$

$$\hat{U}_2(2t) - 2\hat{U}_2(t) = 2U_1(t)\tilde{U}_1(t) + e^{-6\pi i t}U_1^*(t). \tag{16}$$

The solutions are

$$U_1(t) = -\sum_{k=1}^{\infty} \frac{e^{-3\pi i 2^k t}}{2^k}, \tag{17}$$

$$\tilde{U}_1(t) = 0, \tag{18}$$

$$U_2(t) = -\sum_{j,k,l=1}^{\infty} \frac{e^{-3\pi i 2^j (2^{k-1} + 2^{l-1})t}}{2^{j+k+l}}, \tag{19}$$

$$\tilde{U}_2(t) = 0, \tag{20}$$

$$\hat{U}_2(t) = \sum_{j,k=1}^{\infty} \frac{e^{3\pi i 2^j (2^{k-1} - 1)t}}{2^{j+k}}. \tag{21}$$

It is easy to see from Eq. (6) that unstable cycles of length n are

$$\mathrm{Fix}\, f^n = \{z(t_j) : t_j = \frac{j}{2^n - 1}, j = 0, 1, \ldots, 2^n - 2\}. \tag{22}$$

We now evaluate $A_n(D)$ as defined in (4). Note that

$$\begin{aligned}\det Df^n &= \prod_{i=0}^{n-1} \det \begin{pmatrix} 2x_i + \mathrm{Re}(\epsilon) & -2y_i + \mathrm{Im}(\epsilon) \\ 2y_i + \mathrm{Im}(\epsilon) & 2x_i - \mathrm{Re}(\epsilon) \end{pmatrix} \\ &= \prod_{i=0}^{n-1} (4z_i^2 - |\epsilon|^2) \\ &= 4^n (1 - \frac{|\epsilon|^2}{4})^n \prod_{m=0}^{n-1} |z(2^m t_j)|^2,\end{aligned}$$

where the last equality holds to the second order in ϵ. Denote

$$< G(t) >_n = \frac{1}{2^n - 1} \sum_{j=0}^{2^n-2} G(t_j), \tag{23}$$

where t_j's are given by Eq. (22).

$$\begin{aligned}A_n(D) &= \sum_{z \in \mathrm{Fix} f^n} |\det Df^n|^{-D/2} \\ &= 2^{-Dn}(2^n - 1)(1 - \frac{|\epsilon|^2}{4})^{-Dn/2} < \prod_{m=0}^{n-1} |z(2^m t_j)|^{-D} >_n .\end{aligned} \tag{24}$$

Substituting Eqs. (17)-(21) into (11) and using the identity

$$< e^{2\pi i m t} >_n = \begin{cases} 1, & m = 0 \bmod 2^n - 1 \\ 0, & m \neq 0 \bmod 2^n - 1 \end{cases} \tag{25}$$

it can be shown, after some algebra, that

$$< \prod_{m=0}^{n-1} |z(2^m t_j)|^{-D} >_n = 1 + |\epsilon|^2 (\frac{D^2 n}{4} - \frac{Dn}{2} - \frac{D^2}{2} - \frac{Dn}{2^{n+1}} + \frac{D^2 n}{2^{n+3}}), \quad (n > 2). \tag{26}$$

Substituting (26) into (24) yields

$$A_n(D) = 2^{n(1-D)}[1 + |\epsilon|^2 (\frac{D^2 n}{4} - \frac{3Dn}{8})], \quad (n >> 1). \tag{27}$$

If we were to use Eqs. (27) and (2) to obtain a Hausdorff dimension, we would get $D_H = 1 - |\epsilon|^2/(8 \ln 2)$, a value smaller than 1 for small but nonzero ϵ. We show in the following that this value of D_H is incorrect.

Let

$$\chi_n(D) = \sum_{j=0}^{2^n-2} \frac{|z(t_{j+1}) - z(t_j)|^D}{(2\pi)^D}, \tag{28}$$

where $z(t_j) \in \mathrm{Fix}\, f^n$ (Eq. (22)). The Hausdorff dimension D_H of the set $\mathrm{Fix}\, f^n$ in the limit $n \to \infty$ is such that

$$\lim_{n \to \infty} \chi_n(D_H) = 1. \tag{29}$$

This D_H should also be the D_H of the Julia set J. We now evaluate $\chi_n(D)$ to the second order in ϵ. Putting Eqs. (17)-(21) into Eq. (11), we write

$$z(t_{j+1}) - z(t_j) = C_0 + C_1|\epsilon| + C_2|\epsilon|^2. \tag{30}$$

Then to the second order in ϵ,

$$\chi_n(D) = \tfrac{|C_0|^D}{(2\pi)^D}(2^n - 1)\{1 + \frac{D|\epsilon|}{|C_0|^2} < \mathrm{Re}(C_0^* C_1) >_n$$
$$+ \quad \frac{D|\epsilon|^2}{|C_0|^2}[\frac{1}{2} < |C_1|^2 >_n + < \mathrm{Re}(C_0^* C_2) >_n + \frac{D-2}{2|C_0|^2} < (\mathrm{Re}(C_0^* C_1))^2 >_n]\}, \tag{31}$$

where Eq. (23) is used. With the help of the identity (25) we get

$$|C_0|^2 = 2(1 - \cos\frac{2\pi}{2^n - 1}), \tag{32}$$
$$< \mathrm{Re}(C_0^* C_1) >_n \; = 0, \quad (n > 2) \tag{33}$$
$$< |C_1|^2 >_n \; = F(n), \tag{34}$$
$$< \mathrm{Re}(C_0^* C_2) >_n \; = \frac{|C_0|^2}{2}(1 + \frac{1}{2^n}), \tag{35}$$
$$< (\mathrm{Re}(C_0^* C_1))^2 >_n \; = \frac{1}{2}|C_0|^2 < |C_1|^2 >_n, \tag{36}$$

where

$$F(n) = \frac{2}{3} - 2\sum_{k=1}^{\infty} \frac{1}{4^k} \cos 2\pi \frac{3 \cdot 2^{k-1} - 1}{2^n - 1}. \tag{37}$$

The function $F(n)$ can easily be solved for large n in the following way. Note that for $n >> 1$

$$F(n+1) = \frac{1}{2}(1 - \cos\frac{3 \cdot 2\pi}{2^{n+1}}) + \frac{1}{4}F(n)$$
$$= \frac{9\pi^2}{4^{n+1}} + \frac{1}{4}F(n). \tag{38}$$

Substituting $F(n) = H(n)/4^n$ into Eq. (38), we have

$$H(n+1) = 9\pi^2 + H(n), \tag{39}$$

which has the solution

$$H(n) = 9\pi^2 n + a, \tag{40}$$

where a is a constant independent of n. From Eqs. (31), (32) - (36), and (40),

$$\chi_n(D) = 2^{n(1-D)}[1 + |\epsilon|^2(\frac{D}{2} + \frac{D}{4}\frac{< |C_1|^2 >_n}{|C_0|^2})]$$
$$= 2^{n(1-D)}(1 + \frac{9}{16}nD^2|\epsilon|^2), \quad (n >> 1). \tag{41}$$

Eqs. (29) and (41) imply

$$D = 1 + \frac{9}{16\ln 2}|\epsilon|^2. \tag{42}$$

III. THE MAP $f(z) = z^{*2} + \epsilon$

Next, we consider the non-analytic map

$$f_\epsilon(z) = z^{*2} + \epsilon. \tag{43}$$

The map (43) has the property of Eq. (7), so that $J(f_\epsilon) = [J(f_{\epsilon^*})]^*$.

Let us parametrize J in such a way so that

$$f(z(t)) = z(-2t), \quad z(t) \in J. \tag{44}$$

The set of unstable cycles of length n is

$$\mathrm{Fix}\, f^n = \{z(t_j) : t_j = \frac{j}{(-2)^n - 1}, j = 0, \pm 1, \pm 2, \ldots\}. \tag{45}$$

The number of elements in Fix f^n is $|(-2)^n - 1|$. Following similar procedures as in the previous section, we have

$$U_1(t) = -\sum_{k=1}^{\infty} \frac{e^{-2\pi i 4^k t}}{4^k}, \tag{46}$$

$$\tilde{U_1}(t) = -2\sum_{k=1}^{\infty} \frac{e^{-\pi i 4^k t}}{4^k}, \tag{47}$$

$$U_2(t) = -6 \sum_{j,k,l=1}^{\infty} \frac{e^{-2\pi i 4^j (4^{k-1}+4^{l-1})t}}{4^{j+k+l}}, \tag{48}$$

$$\tilde{U_2}(t) = -12 \sum_{j,k,l=1}^{\infty} \frac{e^{-\pi i 4^j (4^{k-1}+4^{l-1})t}}{4^{j+k+l}} + \sum_{k,l=1}^{\infty} \frac{e^{-\pi i (4^k+4^l)t}}{4^{k+l}}, \tag{49}$$

$$\hat{U_2}(t) = -4 \sum_{j,k,l=1}^{\infty} \frac{e^{-\pi i 2^j (4^k+4^l/2)t}}{2^{j+2k+2l}}. \tag{50}$$

$A_n(D)$ (Eq. (4)) and $\chi_n(D)$ ((Eq.(28)) can be calculated to be

$$A_n(D) = \chi_n(D) = 2^{n(1-D)}(1 + \frac{1}{4} n D^2 |\epsilon|^2). \tag{51}$$

In this case, $A_n(D) = \chi_n(D)$ and it gives the correct Hausdorff dimension

$$D_H = 1 + \frac{|\epsilon|^2}{4 \ln 2}. \tag{52}$$

The reason for Ruelle's formula to work in this case is that for the non-analytic map (43) $f^2(z)$ is analytic:

$$f^2(z) = (z^2 + \epsilon^*)^2 + \epsilon, \tag{53}$$

and that $J(f^2) = J(f)$. Note that (52) is the same as the D_H of the analytic map (1) $f(z) = z^2 + \epsilon$ [5,6], to the second order in ϵ. Indeed, $f^2(z)$ and thus J are identical for the two maps (1) and (43) for real ϵ. For complex ϵ, however, the two Julia sets look quite different (Fig. 1).

IV. DISCUSSION

Since Ruelle's formula (2) relies on the analyticity of the map, it is no surprise that it brakes down for non-analytic maps. When J is a closed curve, D_H can be calculated from $\chi_n(D)$ (Eq. (28)) for both analytic and non-analytic maps. When J is no longer topologically a circle, it can be difficult to utilize a formula based on distances between unstable cycle elements. In this case, it remains a challenge to formulate an efficient method for the calculation of D_H for non-analytic maps. Finally, the quantity $A_n(D)$ (Eq. (4)) can be very useful even for non-analytic maps. For example, it can be used to calculate the escape rate for points close to J [7].

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