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Phase Organization

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We describe the selection mechanism for novel pattern formation in a driven chain of nonlinear oscillators. Phase organization arises as a consequence of dynamical selection of minimally stable states.

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Understanding the behavior of complex (many-degrees-of-freedom) systems continues to present a great challenge to physicists. The standard statistical mechanics is a powerful theory for equilibrium and uniform systems; however, for nonequilibrium systems possessing spatio-temporal structure there is no general theory —indeed, it is not even clear what quantities might characterize the physics of such systems. Recently a large step has been taken towards the understanding of these complex systems, in large part because of the study of simple low-dimensionality discrete dynamical equations. It is now realized that (1) simple nonlinear maps can have very rich time series and very complex structures can be produced from very simple algorithms; (2) there exist universality classes, such that very different systems possess identical scaling properties. The hope is that the study of these simple deterministic models can help us pose and answer more general questions in complex systems; this approach is in some sense the obverse of the statistical point of view.

To date, significant progress has been made in understanding complex systems whose behavior can be captured by few "effective degrees of freedom" (EDF)[1]; in contrast, very little is known in general about high-EDF behavior. Systems where few-EDF models appear inadequate include spin-glasses, arrays of coupled maps,[2] spatio-temporal chemical oscillators,[3] neural networks,[4] large-aspect-ratio fluid convention,[5] and a video feedback system.[6] In this Letter, we study a high-EDF system displaying complex dynamics and interesting organizing behavior. In contrast with these other complex systems, ours is simple enough to completely characterize theoretically. Our hope is that this will stimulate progress toward a general characterization of high-EDF behavior.

The model consists of an array of N balls of mass m, each connected to its neighbors by springs with force constant K. The array is subject to a sinusoidal potential and driven by a time-periodic square-wave force $E(t)$. The equation of motion is

$$m\ddot{y} = -\gamma\dot{y}_j + K(y_{j+1} - 2y_j + y_{j-1}) - A\sin(2\pi y_j) + E, \quad j = 1,2,\ldots,N, \qquad (1)$$

where y_j is the position of the jth ball, γ is the damping constant, and A the amplitude of the potential. In the static case ($m = \gamma = E = 0$), Eq. (1) is the Frenkel-Kontorova model which has been studied extensively.[7]

Of particular interest for our purpose is that the Frenkel-Kontorova model has a very large number ($\sim e^N$) of stable configurations—in our problem there is a similarly large number of stable time-periodic stationary states and a hierarchy of metastable states.

What is a phase-organized state? It is a time-periodic state with the property shown in Fig. 1: A snapshot at an instant just before the pulse is turned off shows the majority of balls sitting at maxima of the periodic potential. The phase of ϕ of a ball is its distance from the nearest-potential minimum; a configuration is *phase organized* when $\phi=\frac{1}{2}$ for most of the balls at the falling edge of a pulse. Remarkably, in the appropriate parameter regime (weak springs and large damping), simulations reveal that the phase-organized states are always reached for sufficiently random initial conditions even though these states are measure zero (as N goes to infinity) with respect to the total number of stable states. This phenomenon was first discovered in the context of the pulse-duration memory effect in charge-density waves,[8] where experiments show a macroscopic manifestation of this microscopic organization. We will return to this point briefly later on.

Our goal is to understand the selection mechanism leading to phase organization. We accomplish this by writing down a "Poincaré map" for Eq. (1). By studying the set of fixed points we recognize a "topological" selection for *marginally stable* states and identify the subset of *minimally stable* states as the phase-organized states. The dynamics then selects for these latter states, and reveals the existence of a hierarchy of metastable, "almost" phase-organized configurations.

In the limit of strong damping, weak coupling, and high field ($\gamma \gg m$, $E > A \gg K$), we approximate Eq. (1) by the mapping[9]

$$z_j(n) = y_j(n) + k[y_{j+1}(n) - 2y_j(n) + y_{j-1}(n)] + F \equiv y_j(n) + k\mathcal{D}y_j(n) + F, \tag{2a}$$

$$y_j(n+1) = \mathrm{nint}[z_j(n)], \tag{2b}$$

where $y_j(n)$ and $z_j(n)$ are the positions of the jth ball just before the nth pulse is turned on and off, respectively, and $\mathrm{nint}(x)=\mathrm{int}(x+0.5)$ is the nearest integer of x. The parameters k and F are the effective nearest-neighbor coupling and external force, respectively, and are related to the parameters in Eq. (1).[9] Equation (2b) says that when the pulse is off each ball falls directly into its nearest potential minimum, while Eq. (2a) says that during the pulse the balls are depinned from their minima and their motion is determined by the pulse force F and the nearest-neighbor elastic interaction leading in general to a net drift current. That Eq. (2) captures the essential physics of Eq. (1) is supported by extensive numerical simulations of both the iterative map and the ordinary differential equations; a derivation of a closely related map for $m=0$ is possible within the context of perturbation theory.[10] The chief effect of the approximation leading to Eq. (2) is that simulations of the mapping typically lead to more perfect phase organization than do the corresponding ordinary differential equations. In what follows, we focus on the iterative map since its dynamical properties can be completely characterized by analytic means.

Without loss of generality, we take F to be an integer and rewrite Eq. (2) as

$$y_j(n+1) = y_j(n) + F + \mathrm{nint}[k\mathcal{D}y_j(n)]. \tag{3}$$

Applying the operator $\mathcal{D}$ to both sides and denoting the "curvature" $Dy_j(n)$ as $C_j(n)$ yields

$$C_j(n+1) = C_j(n) + \mathcal{D}\,\mathrm{nint}[kC_j(n)]. \tag{4}$$

Equation (4) is a nonlinear diffusion equation on the *lattice* phase space Z^N. In contrast with linear diffusion, the competition between diffusion (pulse on) and local relaxation (pulse off) allows the chain to stably maintain nonzero curvature configurations. The fixed-point condition $C_j(n+1)=C_j(n)$ implies

$$\mathrm{nint}[kC_j(n)] = m + lj, \quad j=1,2,\ldots,N, \tag{5}$$

where m and l are integers. For periodic or free boundary conditions, only $m=l=0$ is allowed, so that the set of fixed-point solutions is given by

$$-1/2k \le C_j < 1/2k, \quad j=1,2,\ldots,N. \tag{6}$$

Since k is small, this defines a large (compact) hypercube in $\{C_j\}$ space. Any point in or on this cube is an allowed solution[11] and represents a stable time-periodic solution of Eq. (1); it can be proven that there are no attractors outside this cube. The origin is located at the center of the cube and is the most stable configuration, in that it requires the largest (finite) perturbation to violate

FIG. 1. Snapshot of a phase-organized solution of Eq. (1) just before the field is switched off, showing the positions of the balls relative to the sinusoidal potential; $N=55$, $m=1$, $\gamma=100$, $K=1$, $A=100$, $E=305$, $T_{on}=5$, $T_{off}=2$, free boundary conditions. Only an eight-ball portion of the chain is shown.

Eq. (6). Points on the surface of the cube are "marginally stable" (this is a slight misuse of terminology) since at least one of the C_j's only barely satisfies Eq. (6), i.e., $C_j = \pm \text{int}[1/(2k)]$, while points at the corners of the cube are also *minimally stable* since *every* C_j barely satisfies Eq. (6).

Thus, based only on the topology of the solution set, we see that the marginally stable states are preferred. Starting with any initial condition outside the cube, the diffusive nature of Eq. (4) drives trajectories toward the origin, until it hits the cube where it "sticks." One expects that trajectories will hit the surface states with roughly uniform probability, but this is not the case. Rather, the dynamics funnels trajectories toward the corners: If one starts with initial conditions sufficiently far from the cube, the system will arrive at a corner state with probability 1. To understand this selection, one must probe beyond this topological picture, and look at the dynamics outside the cube.

Not only are the corner states minimally stable, but they also correspond to states having perfect phase organization. This follows from Eqs. (2) and (6): One sees immediately that these solutions correspond to $z_j = \frac{1}{2} \pmod 1$ for all j. Thus, each time the pulse is switched off, all the balls are sitting on the tops of the potential! Before giving an explanation for this selection of corner solutions, we note that there is a hierarchy of "almost" fixed points wherein only a tiny fraction of the C_j's change each iteration of Eq. (4). These metastable configurations consist of local domains (subchains) which satisfy Eq. (5) with nonzero m and l, with different domains having different values of m and l. If we consider only the lower-dimensionality phase space for a subchain, the set of possible metastable configurations defines a nested sequence of hypercubes indexed by m. Again, the corners of these nested hypercubes are dynamically preferred.

We now explain the dynamical mechanism underlying the selection of the corner states. Imagine a plot of C_j vs j after each time iteration n (see Fig. 2). Then the stable (metastable) corner states correspond to each C_j being one of the values $\pm 1/2k$ $[\pm(2m+1)/2k]$. Consider a very random initial configuration of C_j's, say C_j distributed uniformly between $-c$ and $+c$ where $c \gg 1/k$. Then for times less than $O(1/k)$, Eq. (4) acts essentially like ordinary diffusion with diffusion constant k, and the shortest wavelength spatial components die out so that neighboring C_j's are likely to have values within $\sim 1/k$ of each other—in fact, it is natural to break the j-C_j plane into horizontal strips such that $2m-1 < 2kC_j < 2m+1$ for each integer m. At this point, the nonlinear function nint in Eq. (4) comes to play an important role, driving each C_j toward the *borders* of these horizontal strips. To see this, suppose that C_{j-1} and C_j are the mth strip with C_{j+1} in the $(m-1)$th. Then $\text{nint}(kC_j) = -1$ and C_j will move down one unit. As long as its neighbors remain in their strips, C_j will continue to move downward until it just enters the $(m-1)$th strip. But now $\text{nint}(kC_j) = +1$, and C_j will move upward one unit, again crossing the border, then down, then up, etc.; so, once C_j reaches the border of a strip, it stays there. By the same argument one easily shows that if C_{j-1}, C_j, and C_{j+1} lie within two neighboring strips, C_j will move to the border of the two strips and stay there. Finally, if the triple are all in the same strip, or in three consecutive strips, C_j will not move. Consequently, there is a tendency for the C_j's to cluster about the border lines of the strips. This is illustrated by the time series shown in Fig. 2. Note that any of these border lines corresponds to phase-organized behavior, though the system has not reached a *bona fide* stable configuration at this stage.

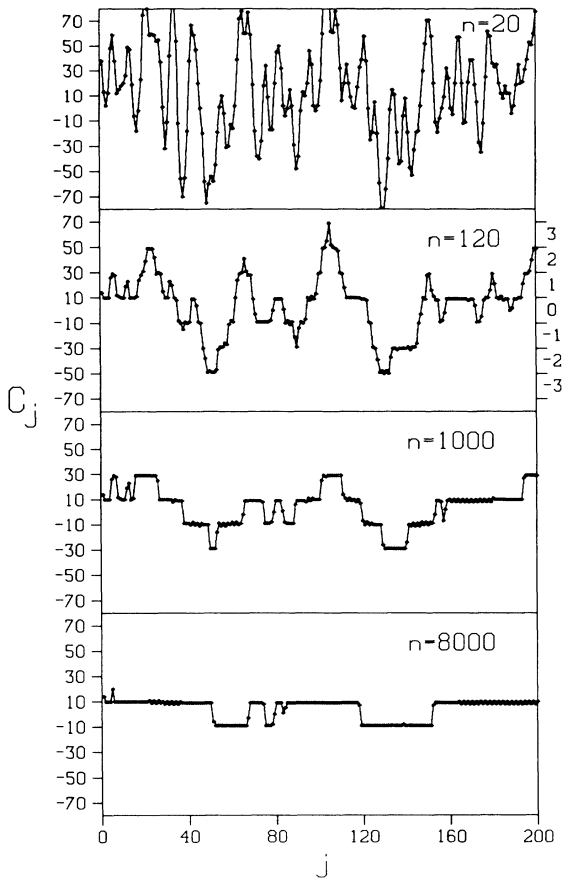

FIG. 2. Curvature C_j of a chain of 200 masses at four times, as Eq. (4) evolves toward a phase-organized state, with $k = 0.05$ and free boundary conditions. From a random initial condition the system quickly evolves toward a metastable state, then slowly relaxes toward a stable corner state. Integers on the right denote the strip index m, as described in the text.

The time scale for this clustering process is $\sim 1/k$ since the width of the strips is $1/k$. After this clustering, the system only very slowly relaxes to one of the stable

corner states, since only the edges of a cluster can have significant change under Eq. (4). This slow "leaking" process towards a stable state takes a time $O(N/k)$.

We now discuss some ramifications of the picture developed above.

First, the fact that the dynamics selects configurations of minimal stability has significant implications for the statistical mechanics of this system. The usual approach would be to seek some global free energy, e.g., the total elastic energy of the chain, and assume a probability density that peaks at the energy minimum. In fact, precisely the *opposite* situation applies here: The most stable (minimum-energy) configuration is the state *least* likely to be observed; the minimally stable (maximum-energy) configurations should be most heavily weighted.

Next, we can extend our analysis to include the effect of quenched randomness modeled by the addition of random constants α_j to Eq. (2a). In fact, it was simulations on this random system that first detected the phenomenon of phase organization.[9] This complication is nonessential to the problem since the rigid translation

$$y_j \rightarrow y_j - \beta_j, \quad z_j \rightarrow z_j - \beta_j,$$

where

$$k\mathcal{D}\beta_j = \alpha_j$$

eliminates the α_j. (This is analogous to the nonessential nature of randomness in the 1D spin-glass problem.[12]) The corner states still are selected, and are phase organized. In effect, the α_j's change the potential from sinusoidal to irregular, but the selected states are still those where the balls sit at potential maxima at the instant the pulse is turned off.

Finally, we revisit the original realization of this model, namely the pulse-duration memory effect in charge-density waves (CDW's). A discussion of the relevance of Eqs. (1) and (2) to CDW's appears in Ref. 9. Recall that periodic voltage pulses cause the CDW to "learn" the length of the pulse. For a "dirty" CDW, the pulses bring the system to some metastable state, so that changing the pulse duration (i.e., changing k in our model) causes the CDW to "relearn" since the structure of the metastable state space has changed. This relearning process has been observed experimentally. However, for a clean enough CDW we can expect a different behavior, since the training pulses may bring the CDW directly to one of the stable corner states. Then, decreasing k (i.e., shortening the pulse) simply expands the stable hypercube, so that the system is already in a stable configuration and no relearning will occur, though increasing k instead will give the familiar effect.

In conclusion, we have studied a periodically pulsed, weakly coupled dissipative many-body system. This high-EDF system displays spontaneous self-organization with a selection mechanism which can be fully characterized. This collective behavior is not due to a long-range correlation (like that in second-order phase transitions), but rather due to a very local dynamical rule; we expect this rule to play a role in a wide class of complex dynamical systems.

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[11]For periodic boundary conditions on y_j, there is an extra constraint on Eq. (6), i.e., $\sum_j C_j = 0$. Thus, the allowed solutions are the intersection of the hypercube with this hyperplane; however, this constraint does not change the general picture for large N.

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